Advanced self-similar solutions of regular and irregular diffusion equations

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/e study various diffusion equations with an appropriate transformation of variables and with the reduction technique. /ith the self-similar and related Ansätze, we reduce the PDE to an ordinary differential equation (ODE). The derived ›lutions of the PDE belong to a family of functions which are presented for the case of the infinite horizon. This study is e second in the line of our investigation. In our first publication, we concentrated on the regular diffusion [1] and presented ımerous solutions which are far beyond the well-known Gaussian or error type functions. Now we supplement the analysis the above mentioned paper with additional generalized self-similar trial functions. We alsoconsider time-dependent ffusion coefficients as well [4]. All of the obtained analytic solutions can be expressed with the Kummer- or Whittaker- pe of functions. These kinds of investigations are organically linked to our long-term scientific activity in which we /stematically examine fundamental hydrodynamic systems [2,3] and analyze physically relevant self-similar and traveling ave solutions. We are also developing numerical methods which are explicit and unconditionally stable at the same time, least for the linear diffusion equation, see e.g. [4, 5] and the references therein.

fter the following change of variables

$$C = \frac{1}{t^{\alpha}} f\left(\frac{x}{t^{1/2}}\right). \qquad (1)$$

the equation of diffusion is transformed as follows

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \quad \rightarrow \quad -\alpha f - \frac{1}{2}\eta\frac{\partial f}{\partial \eta} = D\frac{\partial^2 f}{\partial \eta^2} \qquad (2)$$

where $\eta = x/\sqrt{t}$. The solution of this ODE can be given by the Kummer functions:

$$f(\eta) = \eta e^{-\frac{\eta^2}{4D}}\left[c_1\, M\left(1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right) + c_2\, U\left(1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right)\right] \qquad (3)$$

The most relevant solutions for positive integer values of α can be seen in Fig.1.

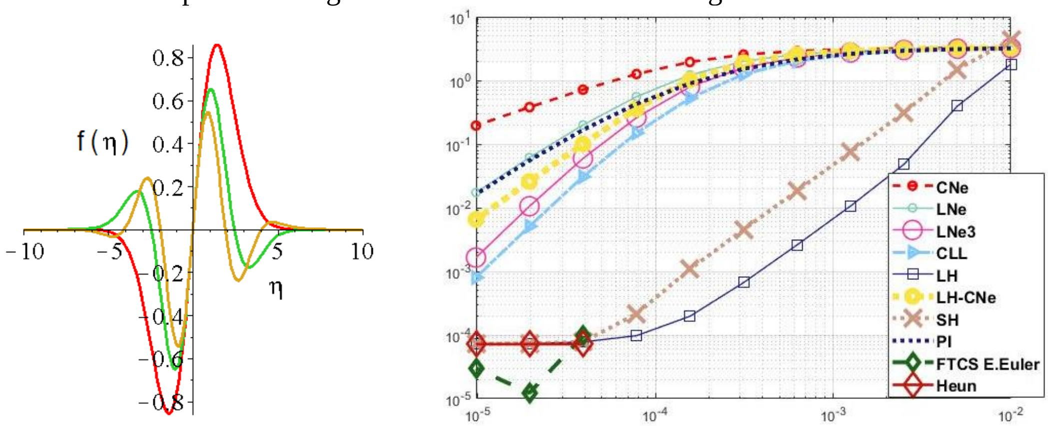

Figure 1: Left: Solutions of the ODE in (3), for α=1 (red), α=2 (green), and α=3 (yellow). Right: Maximum errors as a function of the time step size for 10 different numerical methods.

We transformed back this solution to obtain the solution of the original PDE. Then we reproduced that solution by 8 of our recent methods and by two standard (conditionally stable) methods. On the right side of Fig. 1 one can see the maximum (L_{∞}) errors as a function of the time step size for α=4. The curves with circle markers represent methods which follows the maximum-minimum principle, which is a much stronger property than unconditional stability.

References

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