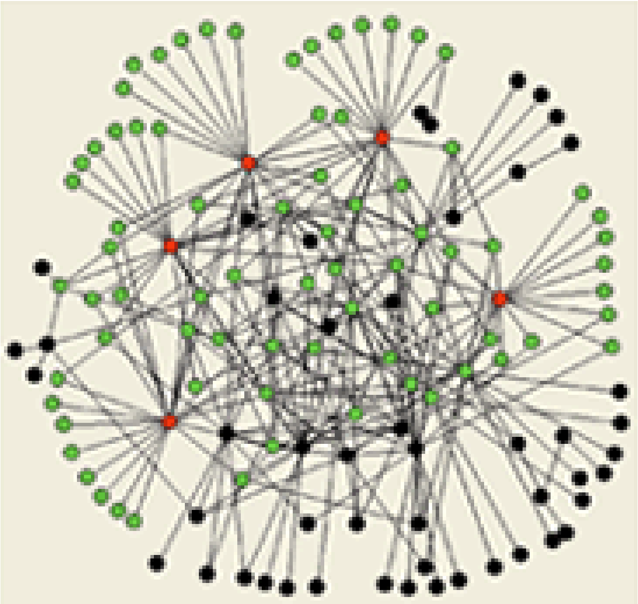
Anomalous Transport in Complex Networks



Shlomo Havlin

Reuven Cohen
Tomer Kalisky
Shay Carmi

Bar-Ilan University

Edoardo Lopez
Gene Stanley

Boston University

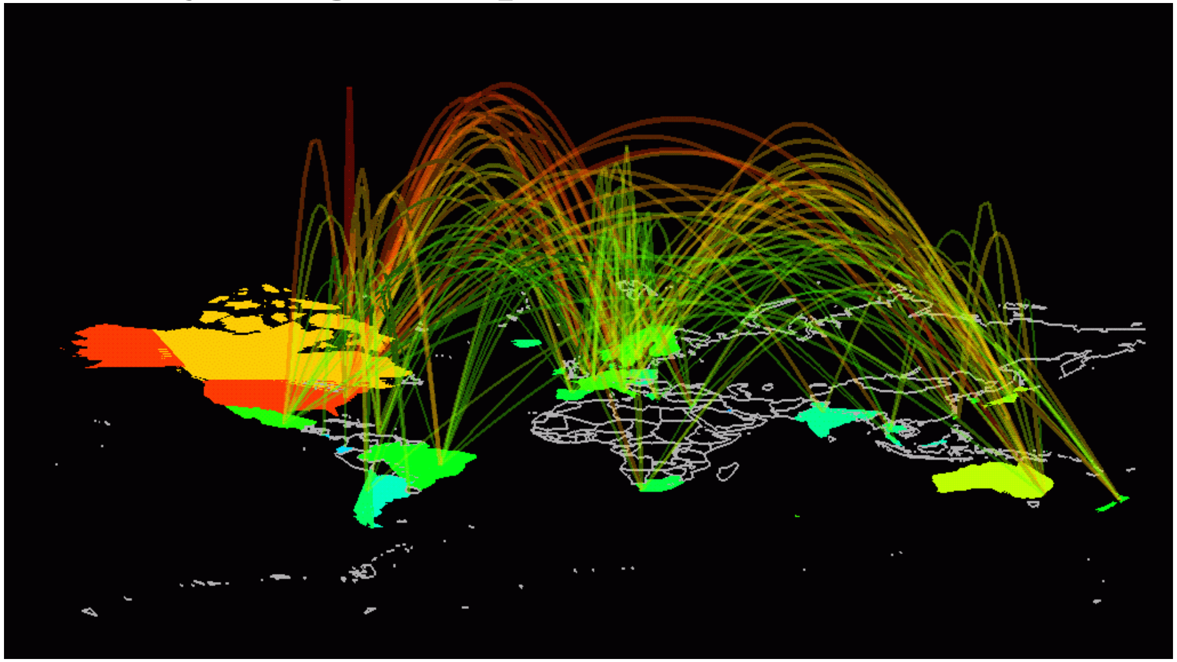
“Anomalous Transport in Scale-free Networks”, López, et al,PRL (2005)

Important Function of Networks -- Transport

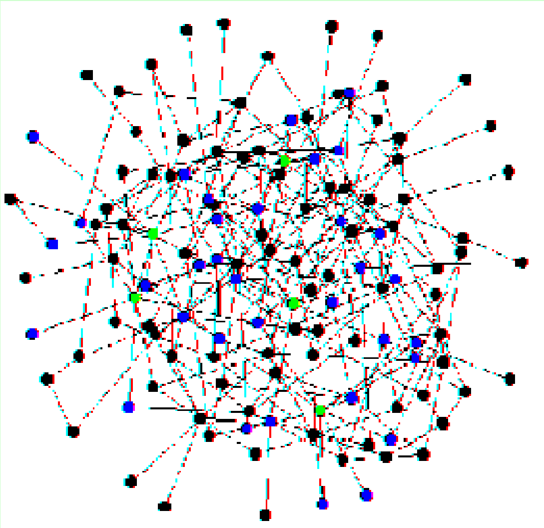
a) Transport: emails, viruses over Internet, epidemics in social networks, passengers in airline networks, etc.

b) Main past focus: studies of ***static properties*** of networks. Robustness, shortest paths, degree distribution, growth models, etc.

c) No general theory of transport properties on networks. Some results-not yet a global picture.



Random Graph Theory



- **Developed in the 1960's by Erdos and Renyi. (Publications of the Mathematical Institute of the Hungarian Academy of Sciences, 1960).**

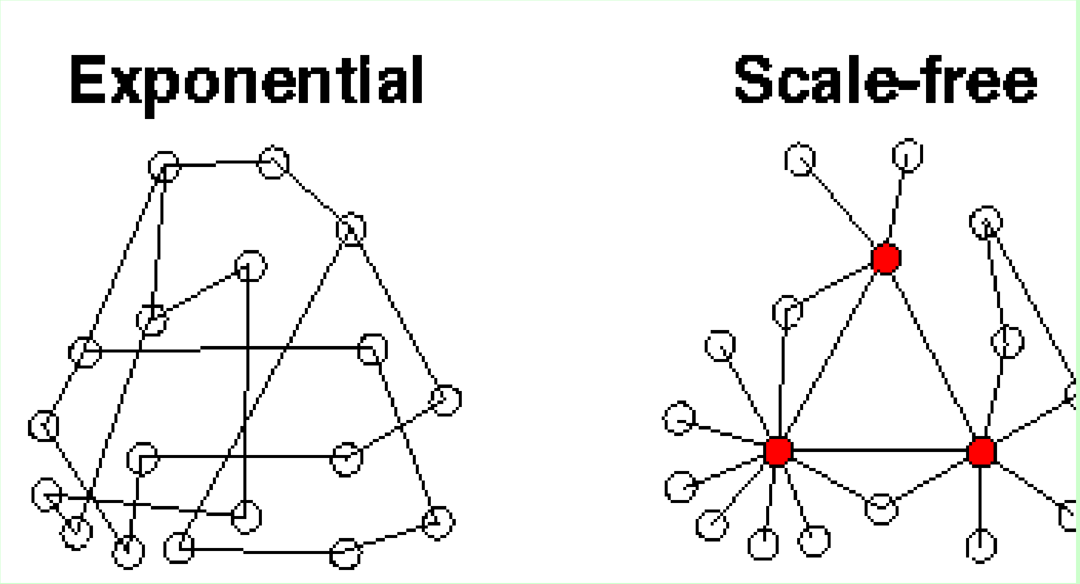
- **Discusses the ensemble of graphs with N vertices and M edges (2M links).**

- **Distribution of connectivity per vertex (degree distribution) is Poissonian (exponential), where k is the number of links :**

$$P(k) = e^{-c}\frac{c^k}{k!}, \quad c = \langle k \rangle = \frac{2M}{N}; p_c = 1 - 1/<k>; \text{MF critical exponents}$$

- **Distance d=log N -- SMALL WORLD**

In Real World - Many Networks are non-Poissonian



$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

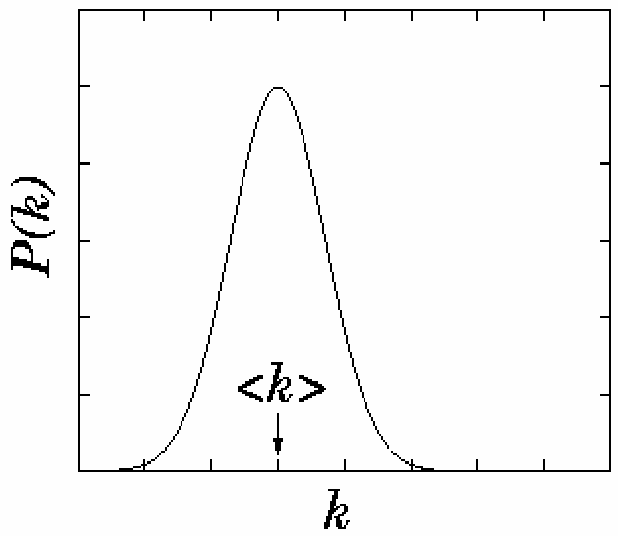
$$P(k) = \begin{cases} ck^{-\lambda} & m \le k \le K \\ 0 & otherwise \end{cases}$$

Erdos-Renyi (1960)

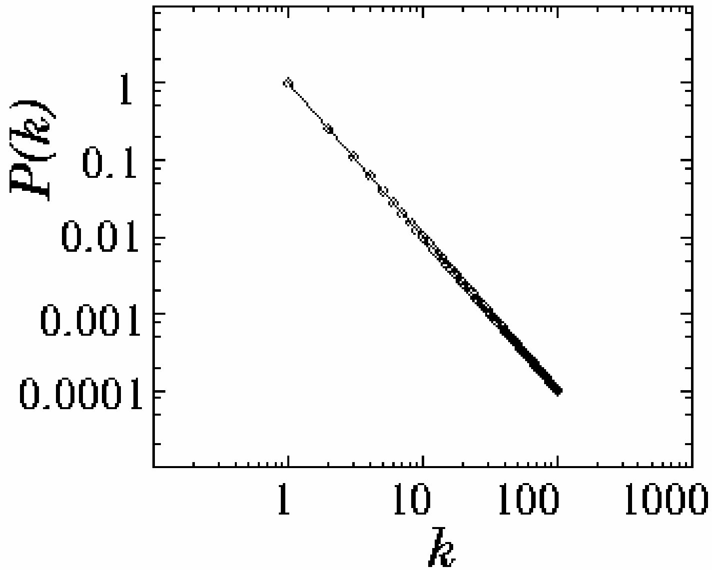
Barabasi-Albert (1999)

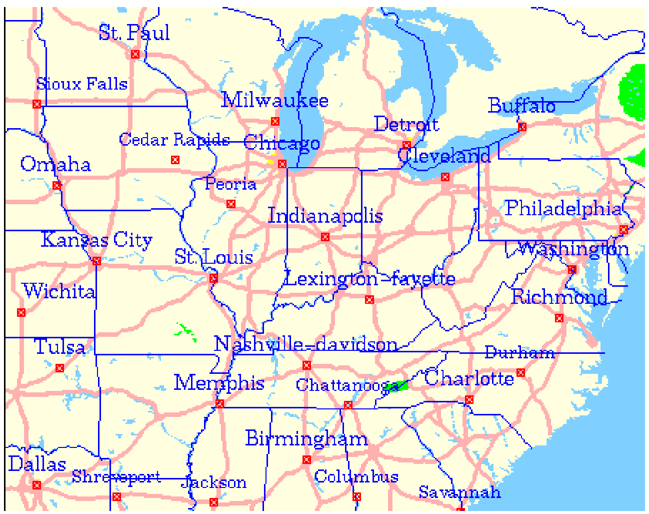
New Type of Networks

Poisson distribution

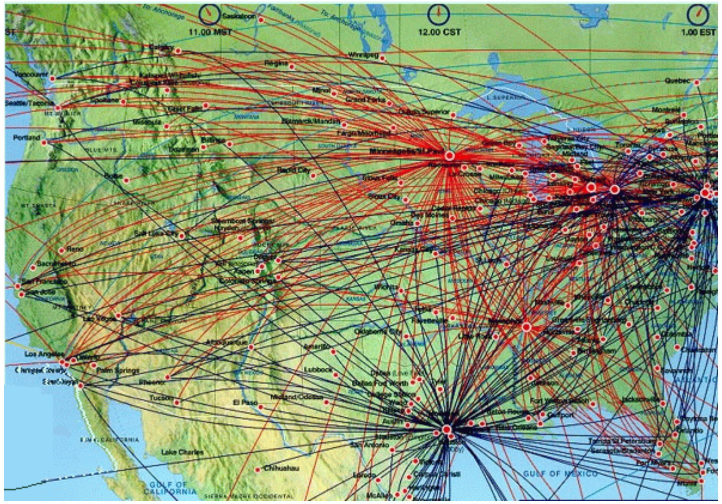


Power-law distribution



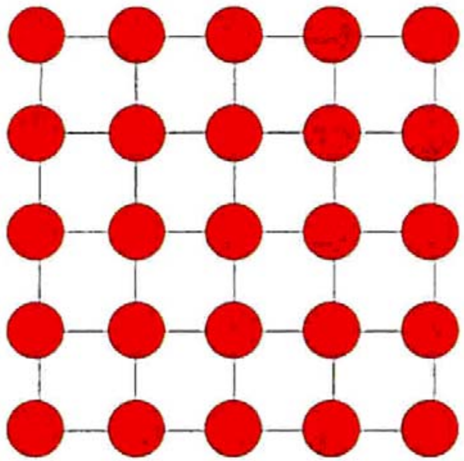


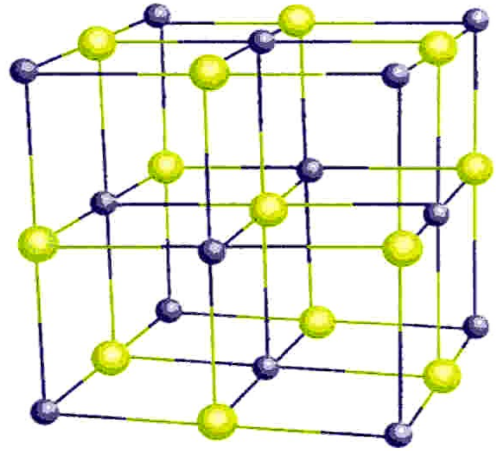
Exponential Network

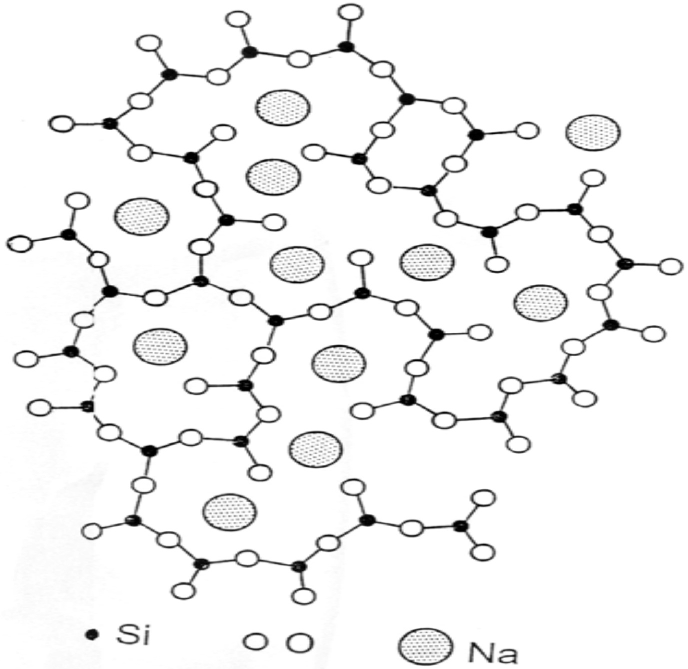


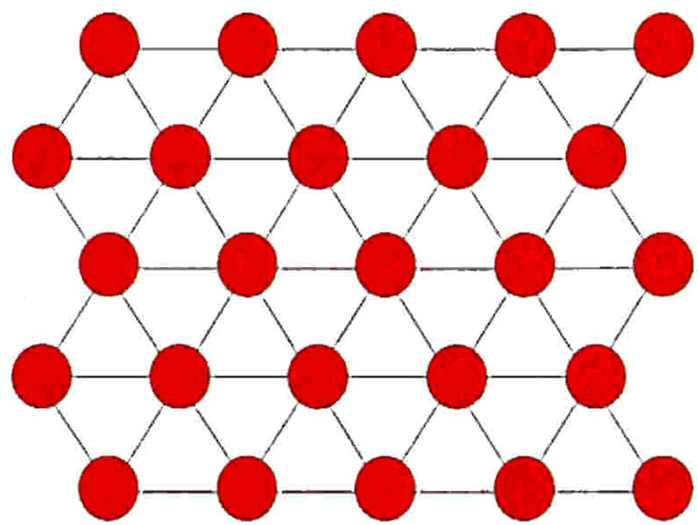
Scale-free Network

Networks in Physics



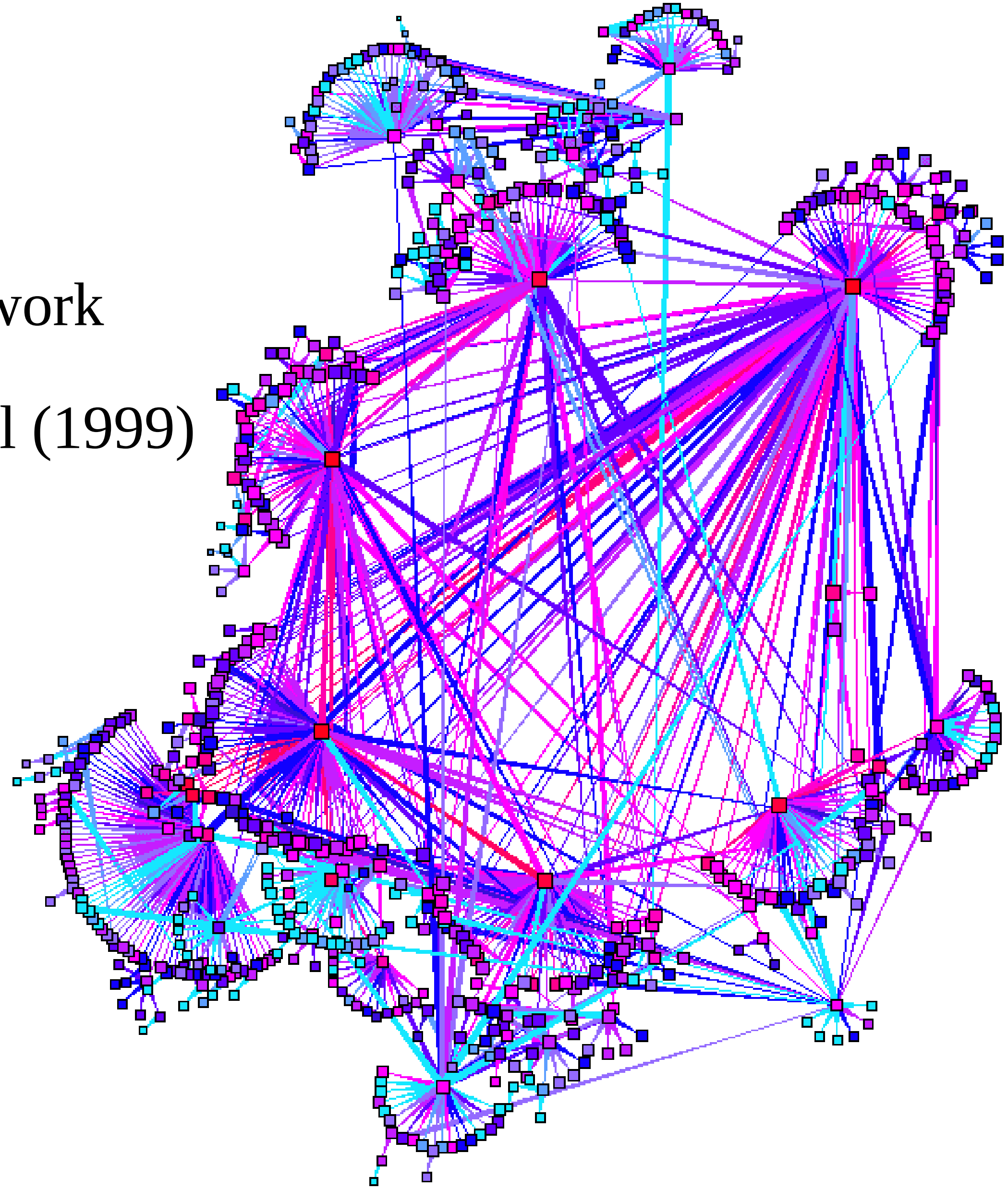






WWW-Network

Barabasi et al (1999)



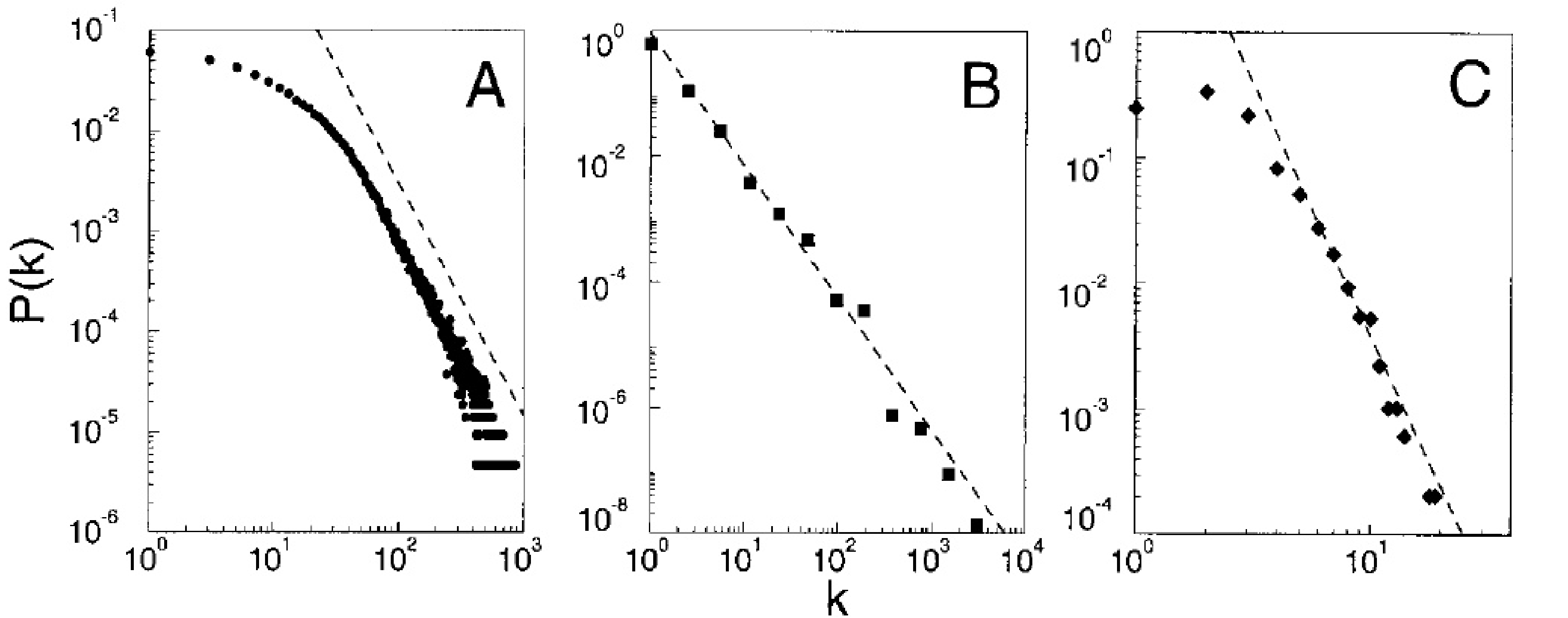
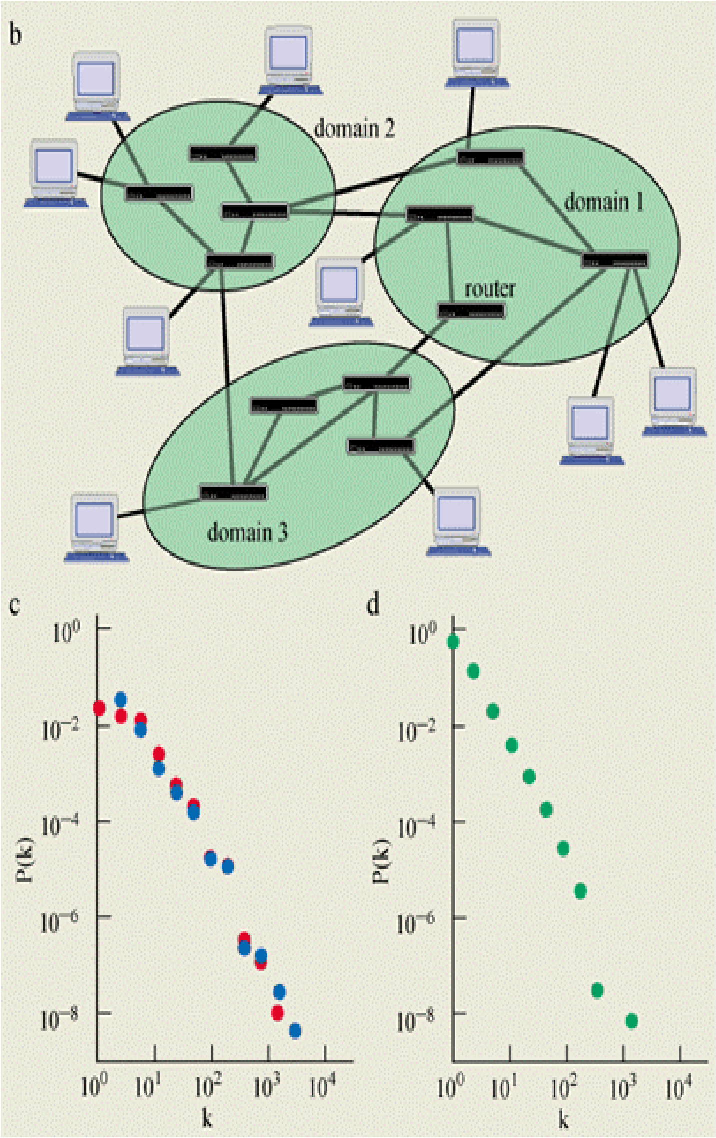


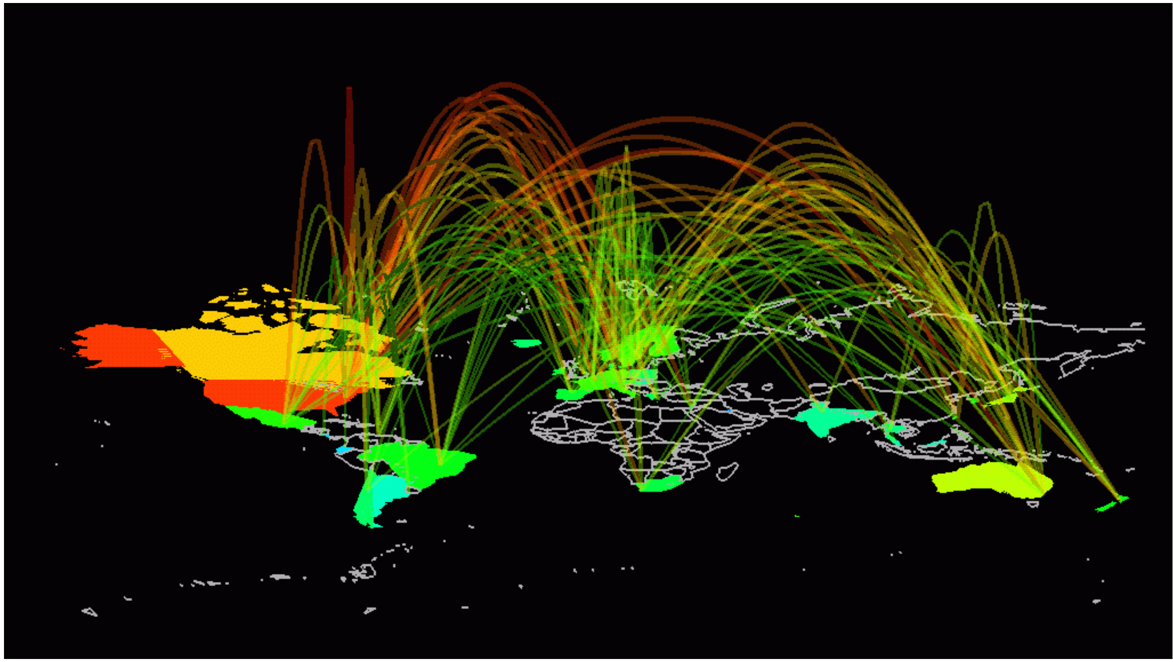
Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

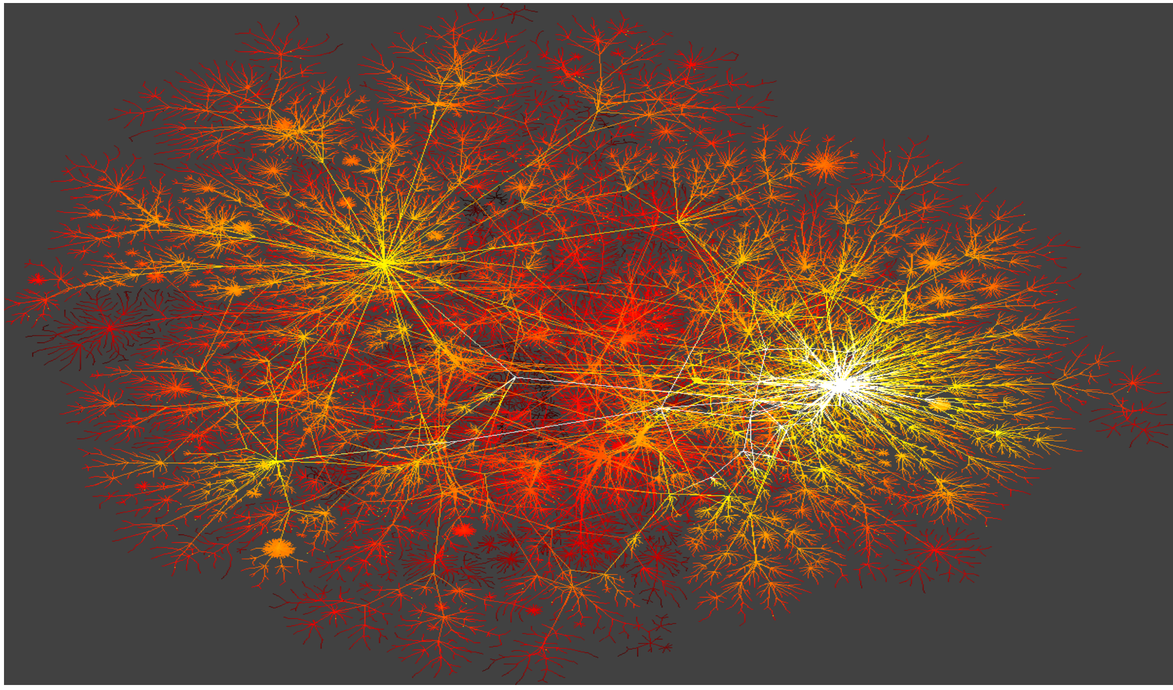
Barabasi and Albert
Emergence of scaling in random networks
Science 286, 509-512 (1999).

Internet Network

Faloutsos et. al., SIGCOMM '99



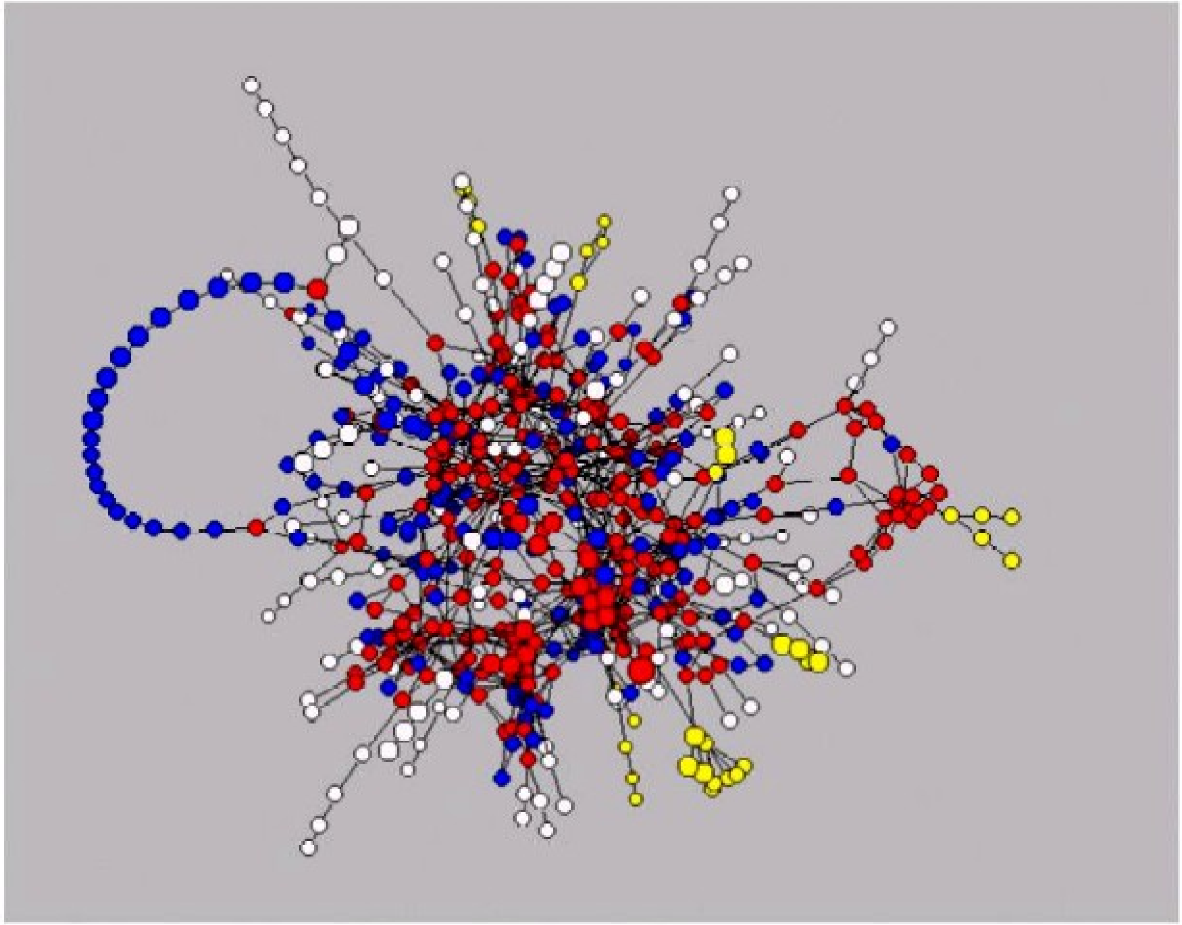




Metabolic Network

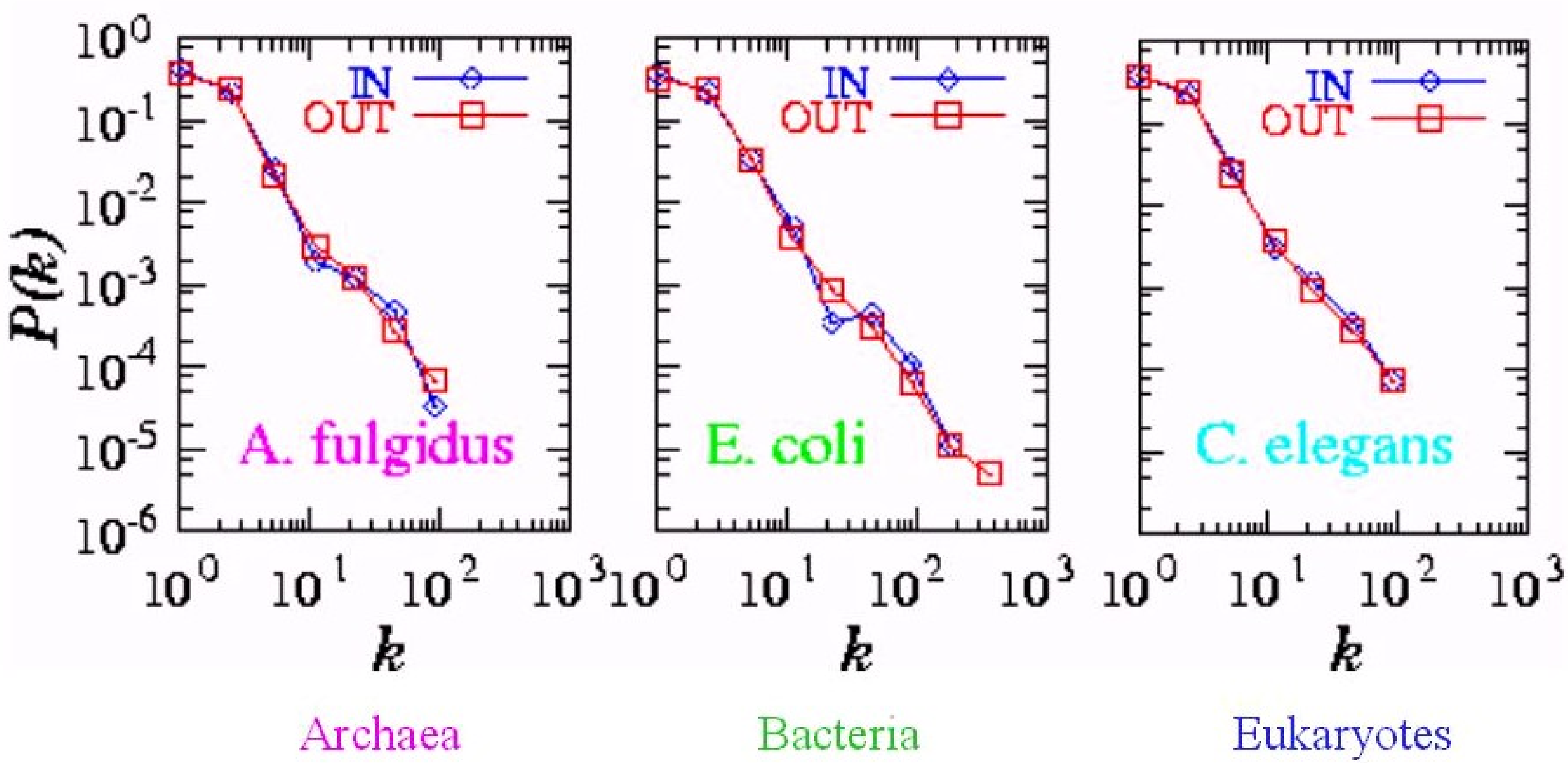
Nodes: chemicals (substrates)

Links: bio-chemical reactions



Jeong, Tombor, Albert, Barabasi, Nature (2000)

Metabolic network



Organisms from all three domains of life are **scale-free** networks!

Distance almost constant does not depend on N

Jeong, Tombor, Albert, Barabasi, Nature (2000)

Many Social networks are also found to be scale free

New models – based on preferential attachment (Barabasi, 2000)
Anomalous Mean Distance in Scale Free Networks

$$P(k) \sim k^{-\lambda}$$

$$\ell = const. \qquad \lambda = 2$$

Ultra Small World

$$\boxed{\ell = \log\log N \qquad 2 < \lambda < 3}$$

$$\ell = \frac{\log N}{\log\log N} \qquad \lambda = 3$$ (Bollobas, Riordan, 2002)

Small World

$$\ell = \log N \qquad \lambda > 3$$ (Bollobas, 1985) (Newman, 2001)

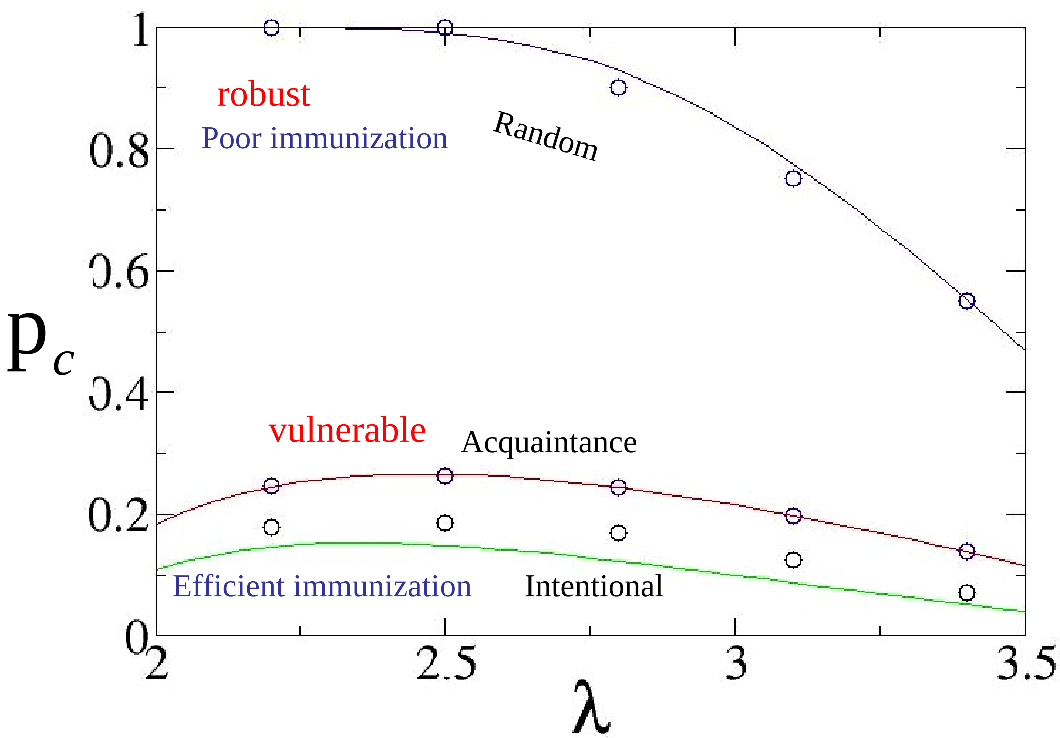
Cohen, Havlin   Phys. Rev. Lett. 90, 58701(2003)

Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks

eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) Chap.4

Confirmed also by: Dorogovtsev et al (2003), Chung and Lu (2002)

More anomalies: Percolation on Scale Free



Cohen et al. Phys. Rev. Lett. 85, 4626 (2000); 86, 3682 (2001**);** 91, 168701 (2003)

General result:

$$p_c = 1 - \frac{1}{K_0 - 1}$$

$$K_0 \equiv \frac{\langle k^2 \rangle}{\langle k \rangle}$$

For Poisson:

$$K_0 = \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle}{\langle k \rangle}$$

$$p_c = 1 - \frac{1}{\langle k \rangle}$$

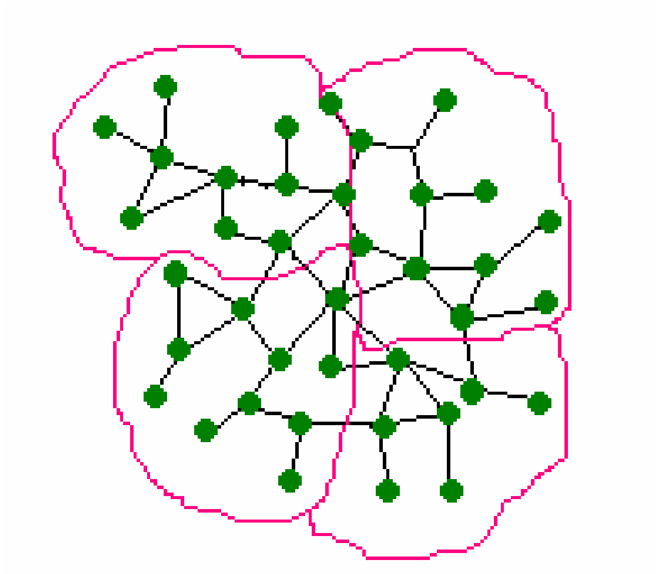
<u>Efficient Immunization Strategie: </u>Acquaintance Immunization

SF new topology→critical exponents are different and anomalous (not regular MF)!

THE UNIVERSALITY CLASS DEPENDS ON THE WAY CRITICALITY REACHED

Scale Free is **not** sufficient: Fractal and Non Fractal Networks

Box counting method



- Generate boxes where all nodes are within a distance ℓ
- Calculate number of boxes, $n(\ell)$, of size ℓ needed to cover the network
- We obtain for WWW, social networks, cellular networks, etc.

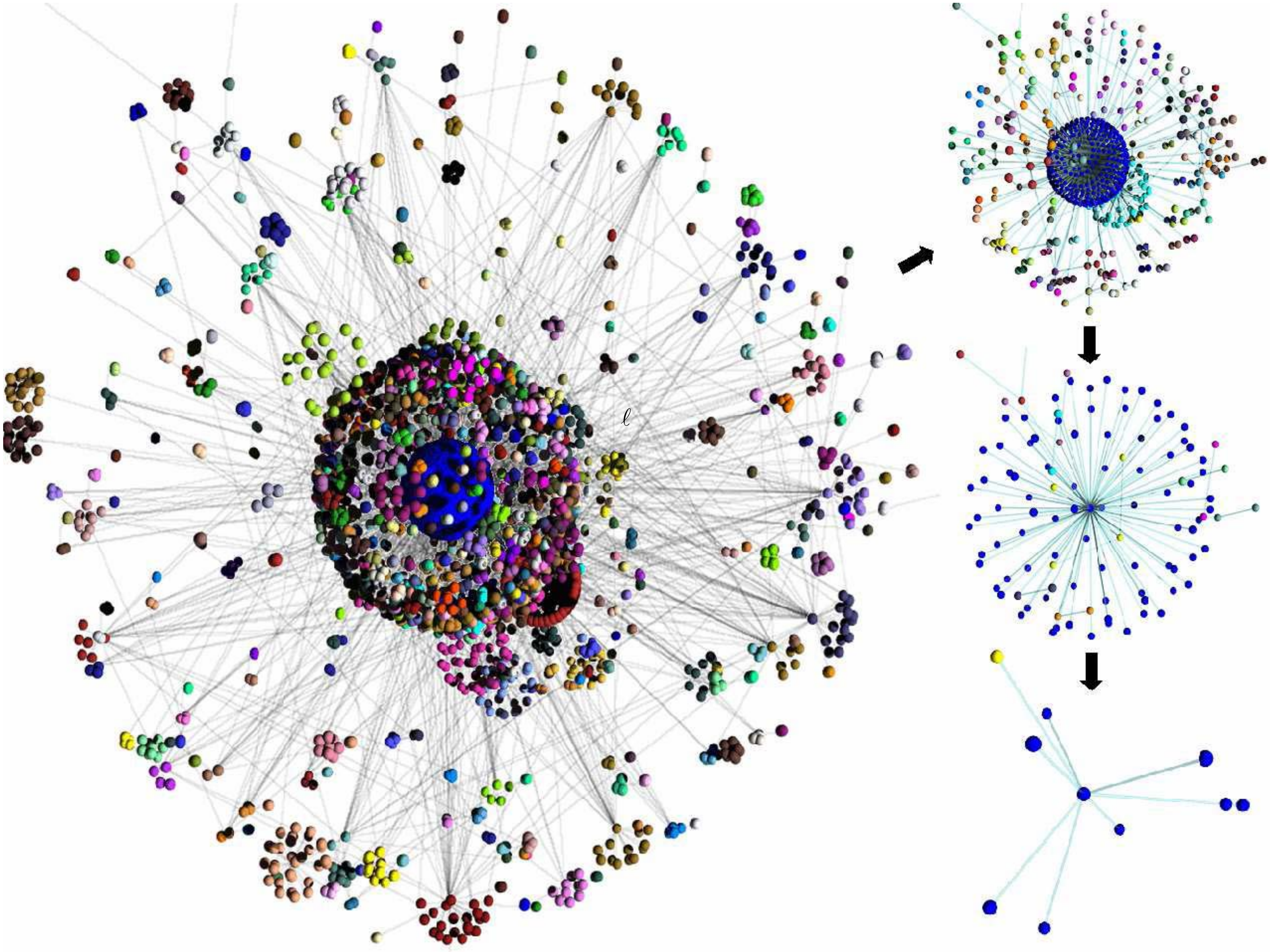
$$N_B(\ell) \square \ell^{-d_B}$$

or

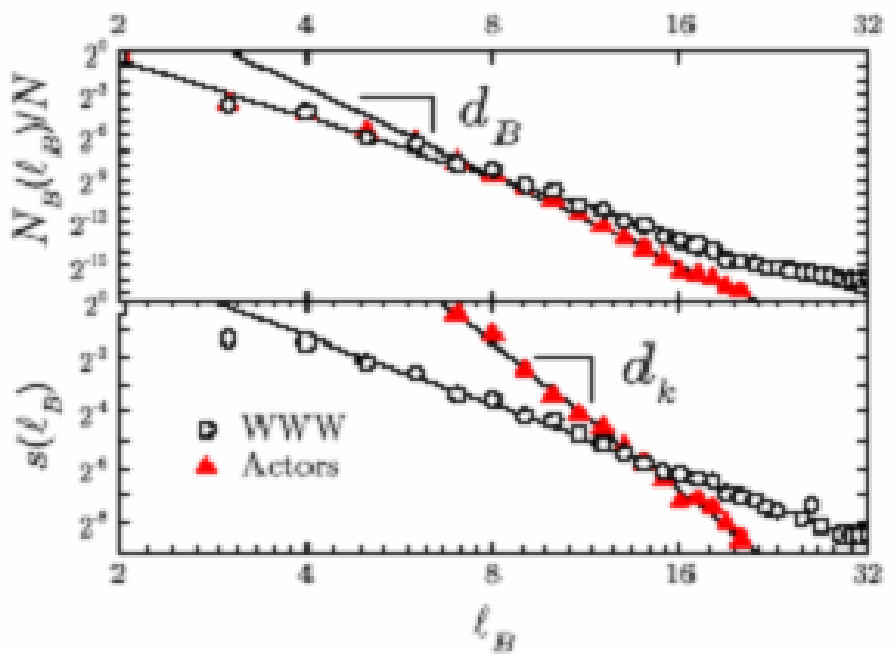
$$M_B \square N / N_B \sim \ell^{d_B} \qquad 2 < d_B < 5 \Longrightarrow \text{Self similarity}$$

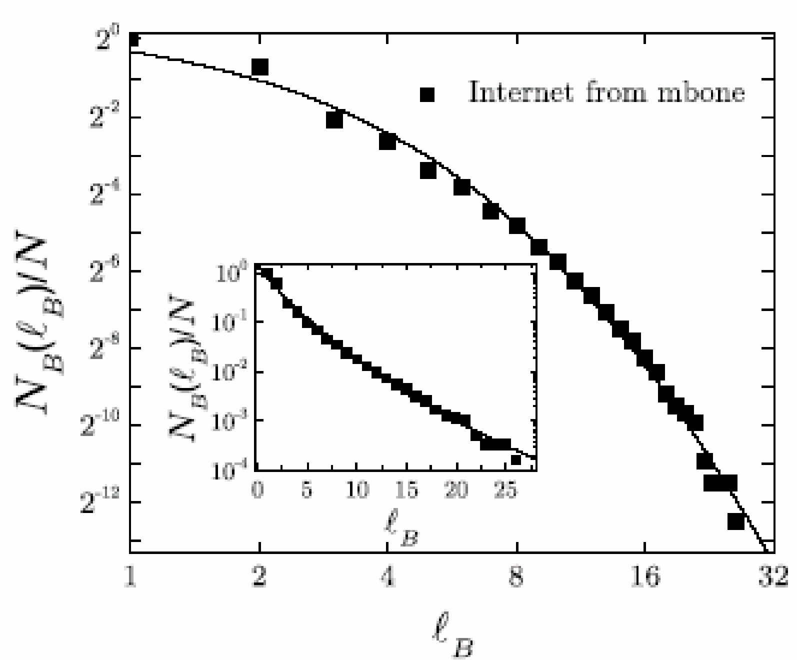
Chaoming Song, SH, Hernan Makse, Nature. 433, 392 (2005); cond-mat/0507216

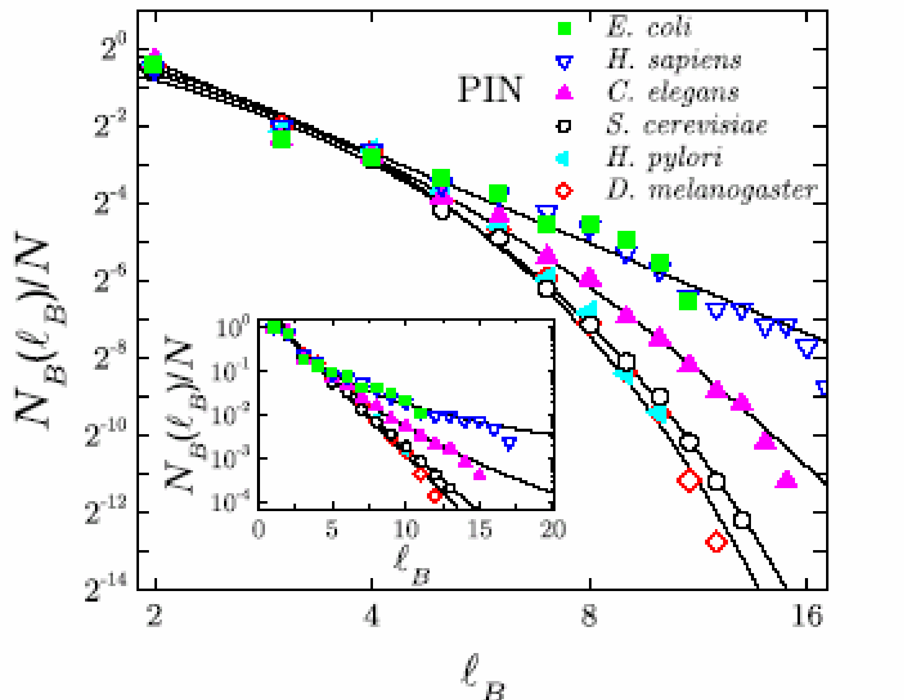
Renormalization of WWW network with $\ell_B = 3$



SOME REAL NETWORKS ARE FRACTALS AND SOME NOT!!







Important Function of Networks -- Transport

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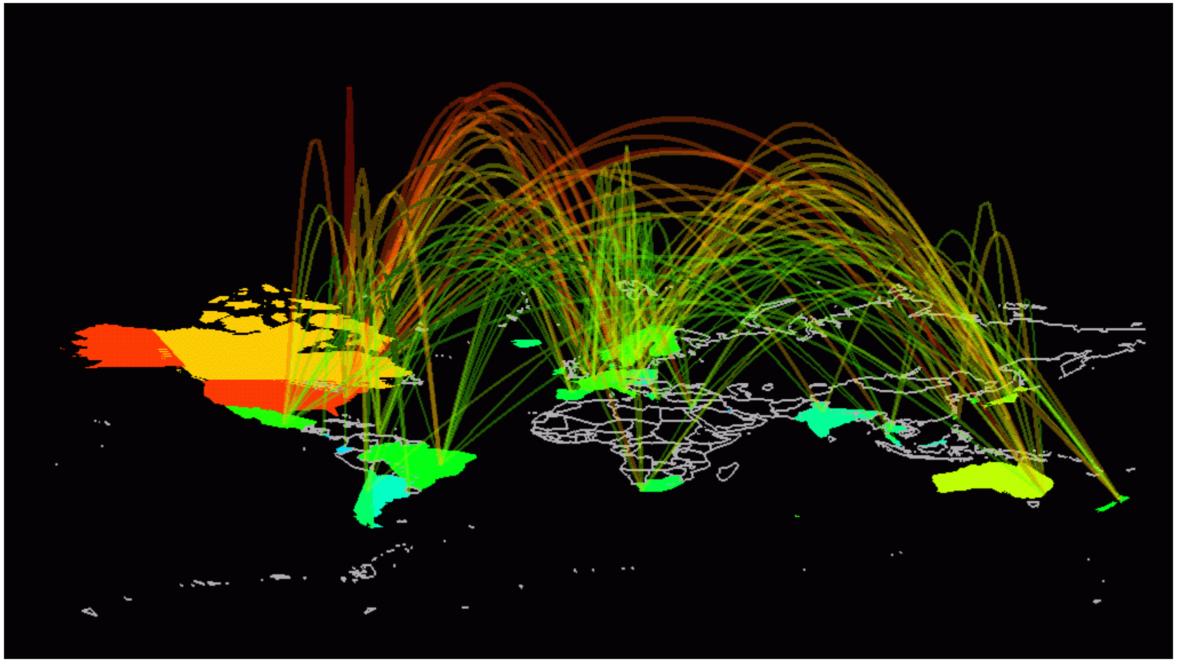
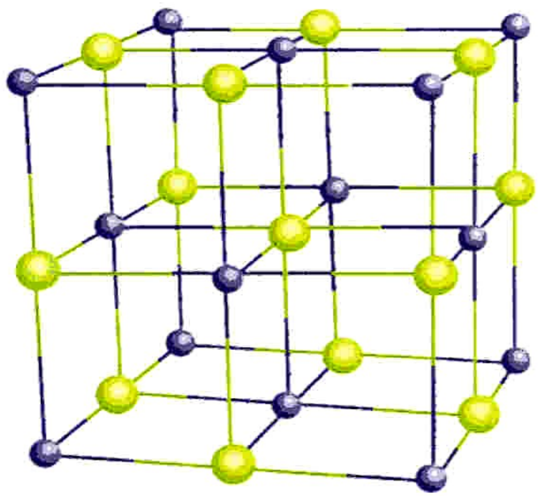
c) No general theory of transport properties on networks. Some important results-not yet a global picture.



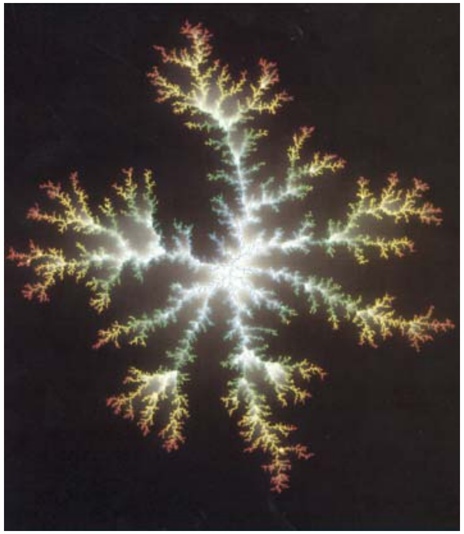
Transport in Complex Networks

Transport on regular networks and fractals:



$$< R^2 >= Dt$$

$$\rho \sim L^{2-d}$$

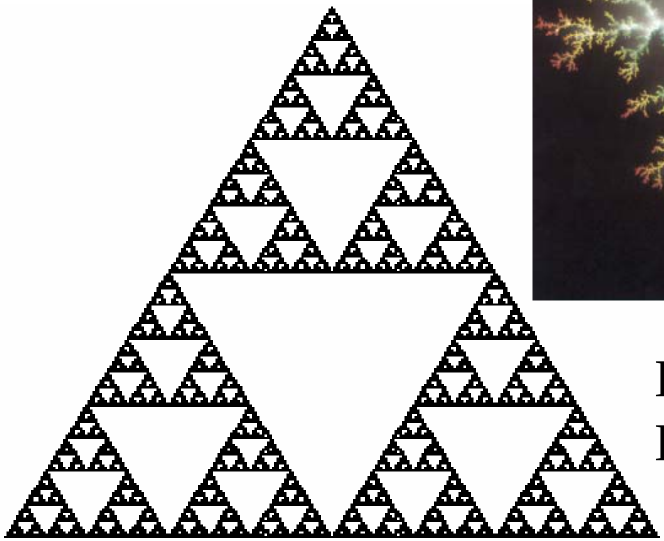


Anomalous transport

$$< R^2 >= At^{2/d_w}$$

$$\rho \sim L^{\zeta}; \; d_w = d_f + \zeta$$

Einstein Relation



Klafter et al,
Levy walks
$d_w < 2$

Bunde-Havlin, Fractals and Disordered Systems, Springer (1996)

Ben-avraham and SH,
Diffusion on Fractals and Disordered Systems, Cambridge Univ. Press (2000)

Transport in Complex Networks

Some important results-not yet a global picture:

1. Bolt and ben-Avraham (NJOP, 2005): transit time faster as the SF network grows; walks are recurrent despite the infinite dimension.

2. Lasaros Gallos (PRE, 2004): super – diffusion $< \ell^2 > \sim n^{2/d_w}$ with $d_w < 2$ depending on λ numerically.

3. Noh and Rieger (PRL, 2004): exact expression for MFPT, $p(\tau) \sim \tau^{-(2-\lambda)}$

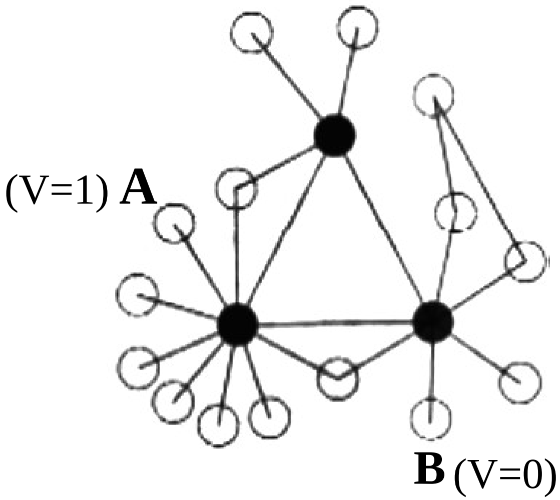
4. Lopez et al (PRL, 2005): Broad distribution of conductances and diffusion constants (depending on degrees)– heterogeneous transport of SF networks

Anomalous Transport in Complex Networks

G – conductance between

two nodes A and B

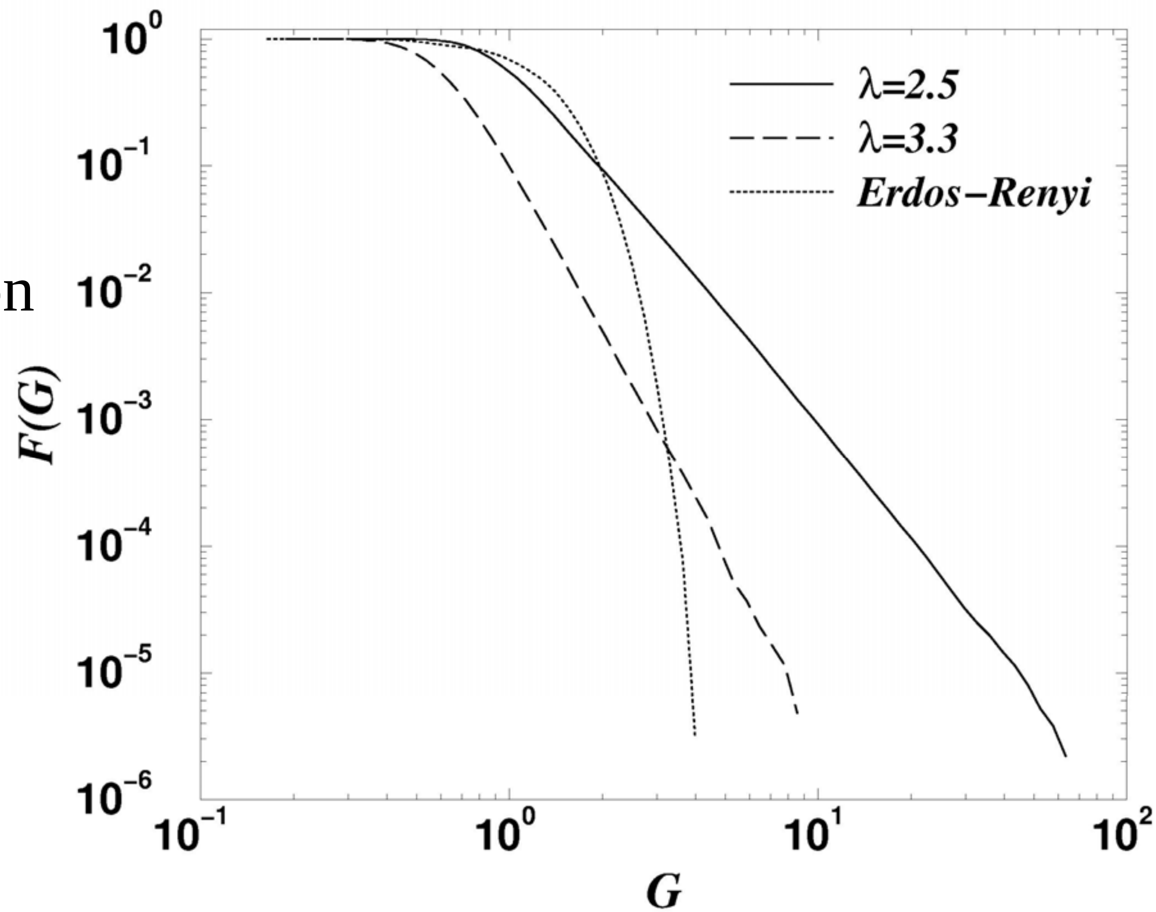
F(G) – cumulative distribution



• each link unit resistor

solving Kirchhoff Eqs•

G ~ D (Einstein relation)



Simulations- N=10,000

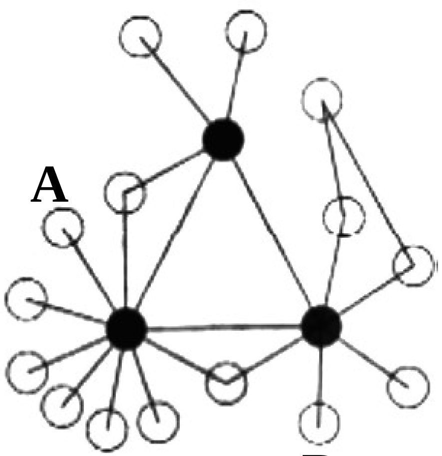
Power law tail

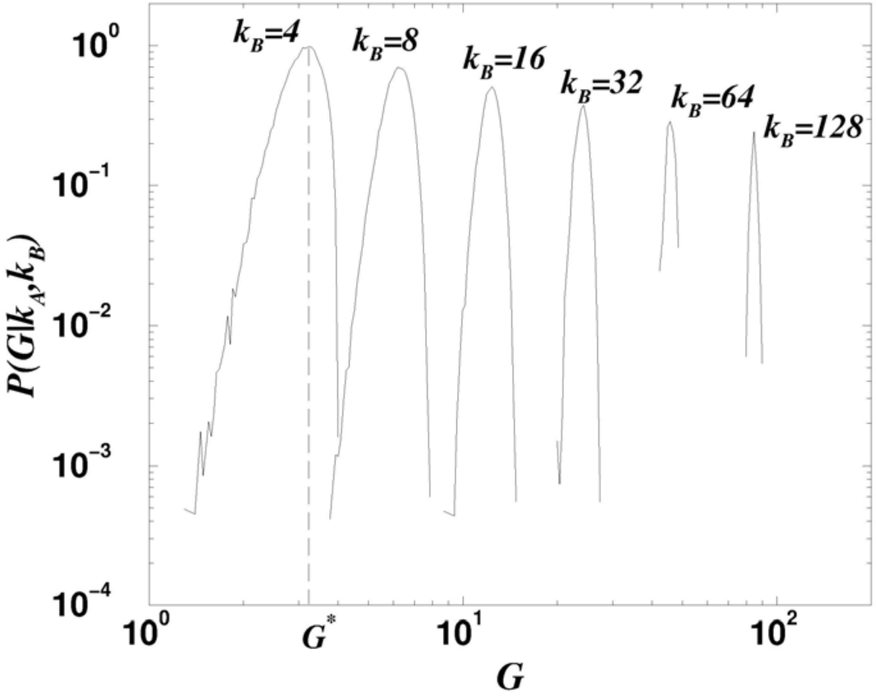
Scale free improve transport

Origin of power law?

$\lambda = 2.5\,,\, k_A \geq 1000$

Strong correlations between conductance and degree of nodes





$P(G\,|\,k_A, k_B)$ - distribution of G given k_A and k_B

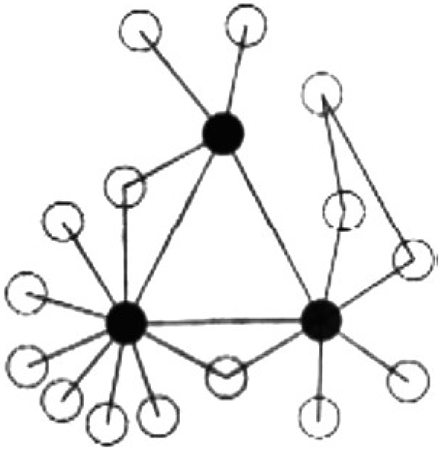
G^* - most probable conductance

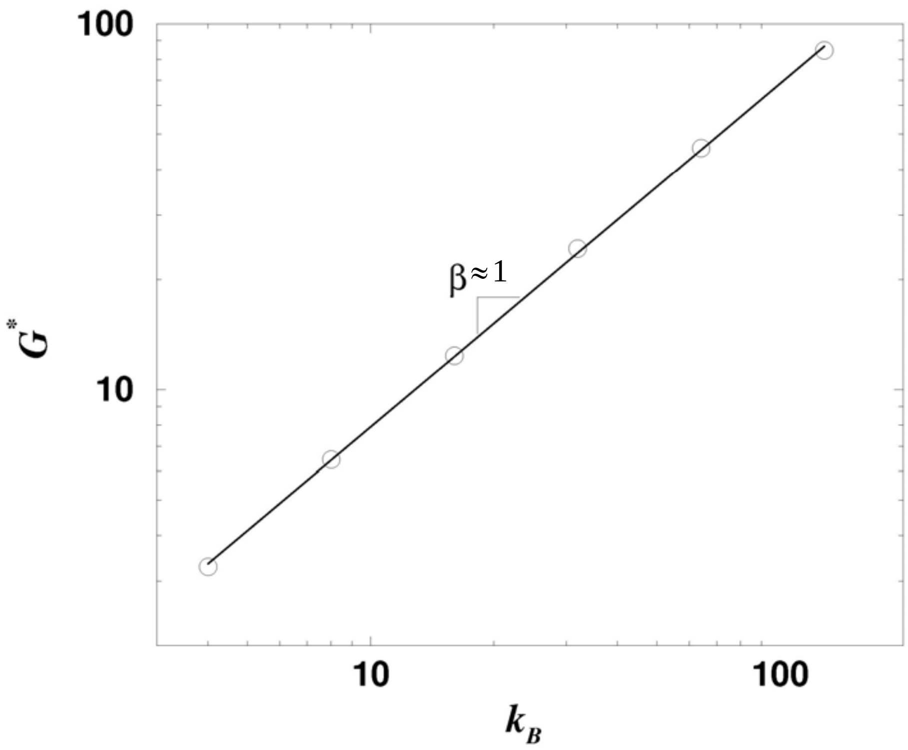
* Large k_A and k_B dominate the high conductanc e regime

* Many parallel paths reduce dramatically the conductance

Origin of power law?

For large k_A $\quad G^* \cong k_B$





$$\Phi(G) = \Phi(k_B) = P(k_B)\int_{k_B}^{\infty} P(k_A)dk_A \approx k_B^{-2\lambda+1}$$

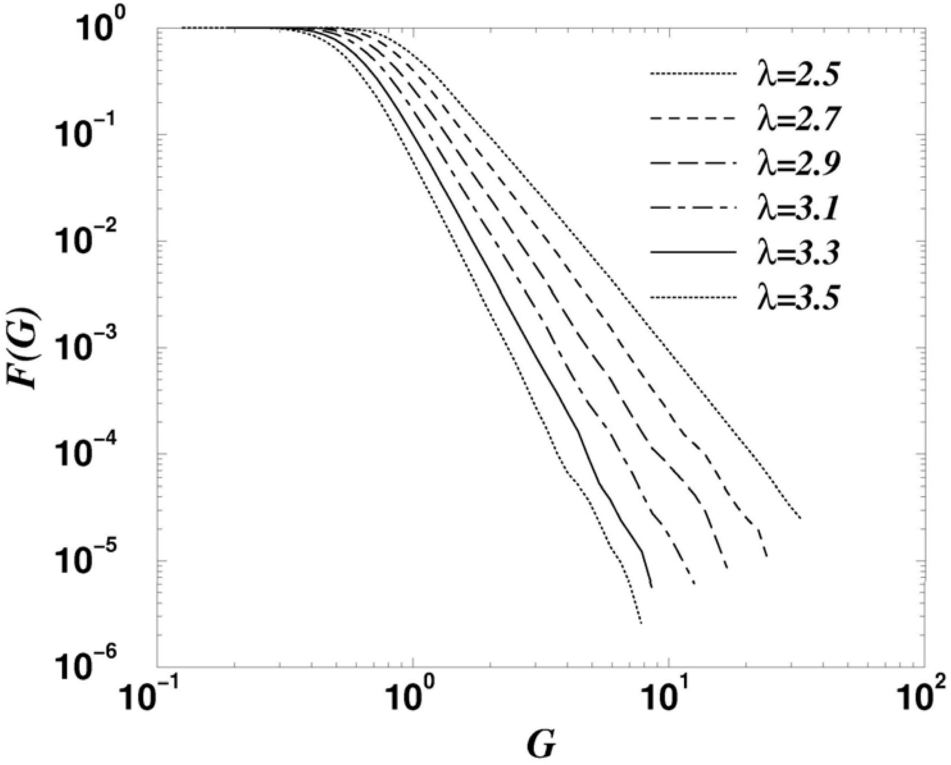
Thus $\Phi(G) \propto G^{-g_G}$ where $g_G = 2\lambda - 1$

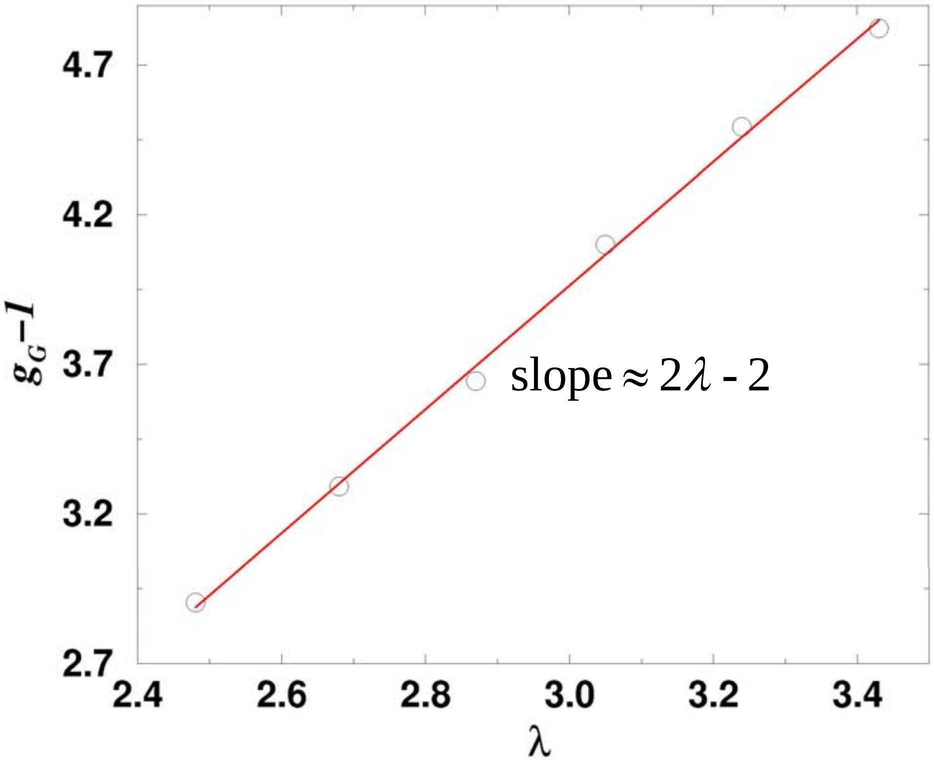
Supported by simulations

Simulations – scale free

$$F(G) = \int_G^\infty \Phi(G)\,dG \propto G^{-g_G+1}$$

where $g_G = 2\lambda - 1$





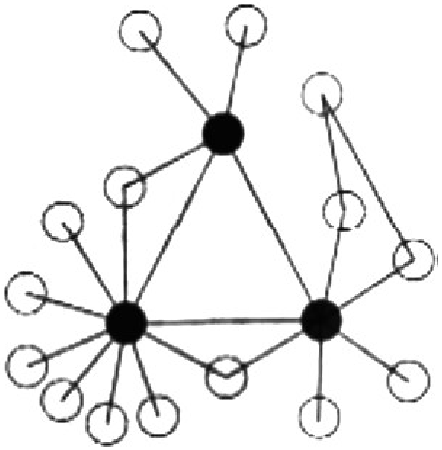
In good agreement with theory

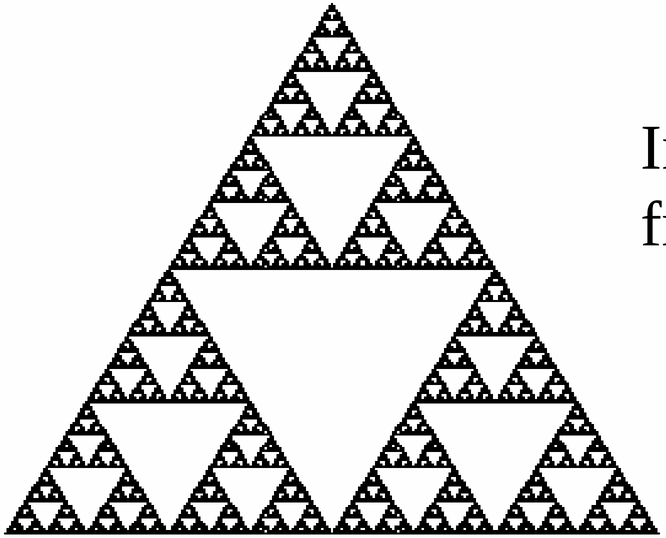
E. Lopez et al, Anomalous transport in complex network, PRL (2005)

Scaling laws of resistance and diffusion for fractal and non-fractals SF networks

$$R(l;k_1,k_2) = l^{\xi} F\left(\frac{k_1}{l^{d_k}}, \frac{k_2}{l^{d_k}}\right)$$

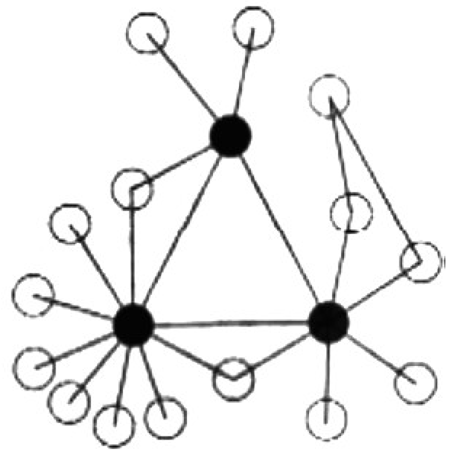
$$t_{walk}(l;k_1,k_2) = l^{d_w} D\left(\frac{k_1}{l^{d_k}}, \frac{k_2}{l^{d_k}}\right)$$





In regular homogeneous fractals D and F are constants

Conclusions and Applications



Scale Free - $p(k) \sim k^{-\lambda}$:

* Anomalous properties- d=loglog N, diff. percolation
* Rich topology: Fractal-Nonfractal real networks
* Generalization of ER: $\lambda > 4$ ER, Infinite dimension, regular MF

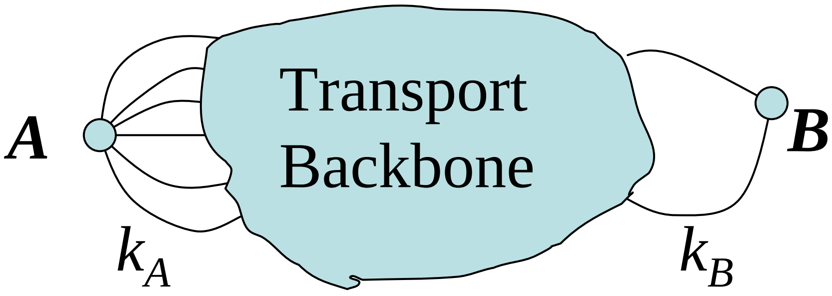
Anomalous Transport:

* Broad distribution of diffusion constants or conductances
* Heterogeneous {Dij} depending on nodes (i,j)-mainly on degree- due to heterogeneous topology.

Applications:

* Optimize topology of networks against various types of failures
* Optimize transport, searching and navigating in networks

Simple Physical Picture



• Network can be seen as series circuit.

•Conductance G^* is related to node degrees k_A and k_B through a network dependent parameter c.

•To first order (conductance of “transport backbone” >> ck_Ak_B)

$$G^* = c\,\frac{k_A k_B}{k_A + k_B}$$